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Quantum mechanical models in fractional dimensions

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Abstract

We formulate an algebraic approach to quantum mechanics in fractional dimensions in which the momentum and position operators P, Q satisfy the R-deformed Heisenberg relations, which depend on an operator ν. We find representations of P, Q in which the dimension d and angular momentum ℓ appear as parameters related to the eigenvalues of ν. We analyse the domain of P and find conditions which ensure that P is Hermitian. We investigate plane wave solutions and also free particle wavefunctions in fractional dimensions, and show that as a consequence of wavefunction continuity ℓ is quantized. The representations of P, Q also lead to the corresponding representations of paraboson operators which are used to solve the harmonic oscillator in dimension d, both algebraically and analytically. We demonstrate that the formalism extends also to time-dependent Hamiltonians by solving the time-dependent harmonic oscillator in any dimension $d > 0$ using the method of Lewis and Riesenfeld.

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1. Introduction

The possibility that some phenomena in quantum physics may be accurately modelled in spaces of fractional dimensions have been regularly discussed in the literature. The putative fractional dimension may be viewed as an effective dimension of compactified higher dimensions or as a manifestation of a non-trivial microscopic lattice structure of space (Schäfer and Müller [1], see also [2]). In some applications, the fractional dimension appears as an explicit parameter when the physical problem is formulated in d dimensions in such a way that d may be extended to non-integer values, as occurs in Wilson's study of quantum field theory models in less than four dimensions [3], or in the approach to quantum mechanics by Stillinger [4]. Other examples (taking a selection only) are the modelling of excitons in 2D or 3D anisotropic solids (He [5]), in exciton–phonon interactions (Thilagam [6]), in the explanation

of absorption structures of reduced dimensionality (Tanguy, Lefebvre, Mathieu and Elliot [7]), and superconductors exhibiting a multilayered structure [8]. For a discussion of 'mesoscale' systems with characteristics of fractional dimensions, see [9].

In the approach by Stillinger [4], followed by He [5], the Schrödinger equation is solved in dimension d, for central potentials $V(r)$ only, by first writing the d-dimensional Laplacian Δ in the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0 \tag{1}$$

where r is the radial coordinate and Δ_0 is the Laplace operator on the unit sphere S^{d-1} (see Vilenkin [10], chapter IX). The Schrödinger equation splits into radial and angular differential equations with the latter having solutions (the eigenfunctions of Δ_0) that are determined by the d-dimensional rotation group, and are independent of the form of the central potential. Since Δ_0 has eigenvalues $-\ell(\ell+d-2)$, where ℓ is a non-negative integer, the Laplacian Δ may be replaced by the following operator, the radial Laplacian,

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+d-2)}{r^2} \tag{2}$$

which acts only on radial wavefunctions and is defined now for non-integer values of $d > 0$. One therefore solves the equation

$$[-\Delta_r + V(r)]\psi = E\psi \tag{3}$$

for the wavefunctions ψ and energy levels E, and the corresponding Hilbert space consists of radial functions $\psi(r), \phi(r)$ with an inner product defined by

$$(\psi, \phi) = \sigma_d \int_0^\infty r^{d-1}\overline{\psi(r)}\phi(r)\,\mathrm{d}r.$$

The constant σ_d may be given by

$$\sigma_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

in order to correctly reproduce the volume of a radius-R ball in d dimensions:

$$B_d(R) = \sigma_d \int_0^R r^{d-1}\,\mathrm{d}r = \frac{\pi^{\frac{d}{2}}R^d}{\Gamma\left(1+\frac{d}{2}\right)}.$$

This formulation of the Schrödinger wave mechanics in fractional dimensions, discussed in detail by Stillinger [4] and used also by other authors [11], allows the dimension d to take non-integer values and underlies several of the fractional-dimensional models considered above. Despite the disappearance of rotational symmetry (since we now ignore the angular equations), the angular momentum ℓ is nevertheless restricted to integer values which follow from the required behaviour of the wavefunctions near the origin. This approach to fractional-dimensional models is, however, restricted to Hamiltonians which have the form $H = -\Delta_r + V(r)$, with a central potential $V(r)$, and neglects the degeneracy of the spectrum due to the d-dimensional rotational symmetry of H. Hence, the wavefunctions do not depend, for example, on the magnetic quantum number m. A formulation of the quantum mechanics in fractional dimensions with more degrees of freedom is also possible but will be considered elsewhere.

We analyse here the properties of several models using an algebraic approach to quantum mechanics in fractional dimensions which reproduces the features of the dimensional extension discussed above. Our approach, which is outlined in section 2, is restricted to one degree of

freedom, and so the algebraic formulation involves a single momentum operator P, which may be viewed as the square root of the d-dimensional radial Laplacian operator Δ_r, and a position operator Q which is represented by the radial coordinate r. As a consequence, the models we consider allow only central potentials and the wavefunctions are independent of the magnetic quantum number m, as before, although they do depend on the angular momentum ℓ. In section 3, we describe representations of the defining algebraic relations showing in particular how the physical quantities d, the dimension, and ℓ, the angular momentum, appear within these representations. In section 4, we derive conditions under which the operators P, Q are Hermitian.

The algebraic formulation has several advantages over the purely analytical approach, for example, we can in principle consider Hamiltonians $H(P, Q)$ with a general dependence on P, Q, and we are also able to take full account of any symmetries and Lie algebraic properties of the model, and hence explain any degeneracies of the energy levels. In this paper, we restrict our attention to elementary models considering firstly eigenfunctions of the momentum operator and free-particle wavefunctions (section 5). Then we analyse the harmonic oscillator in d dimensions, reproducing firstly previously known results but then generalizing them to include all values of angular momentum. In order to demonstrate that our approach is valid for more general models, we also consider the time-dependent harmonic oscillator in fractional dimensions, for which the frequency $\omega(t)$ is a given time-dependent function, and find all wavefunctions using the method of Lewis and Riesenfeld (section 7).

Our algebraic approach also allows us to construct coherent and squeezed states (which can be time dependent) in fractional dimensions and also to develop properties of Weyl-ordered polynomials, which are useful as a means of constructing various quantum mechanical operators [12].

2. Algebraic formulation

The algebraic formulation of fractional-dimensional quantum mechanics involves a third operator R, the reflection operator, which together with P, Q, satisfies the relations of the following R-deformed Heisenberg algebra:

$$[Q, P] = \mathrm{i}(1 + \nu R) \qquad \{Q, R\} = 0 = \{P, R\} \qquad R^2 = 1 \tag{4}$$

where ν is a real number although, as we show, should actually be regarded as an operator which commutes with each of P, Q, R. We also have the hermiticity properties

$$Q^* = Q \qquad P^* = P \qquad R^* = R.$$

In addition to the usual invariance with respect to linear $\mathfrak{sl}_2$ transformations, discussed further in [12], these relations are also invariant under the discrete symmetry

$$R \longrightarrow -R \qquad \nu \longrightarrow -\nu. \tag{5}$$

The relations (4) are not invariant under translations, $Q \to Q + \lambda$ for nonzero $\lambda \in \mathbb{R}$, which can be understood if Q is represented as a radial coordinate since translations act directly on Cartesian, not radial, coordinates; this is consistent with our fractional-dimensional interpretation. It implies also, for example, that models with potentials $V(Q)$ have origin-dependent properties that depend on the precise form of V.

The algebra (4) appeared in the work by Vasiliev [13] in 1989 but may be derived from relations postulated by Wigner [14] in 1950, in which he considered the possibility that the quantum mechanical commutation relations need not coincide with the canonical commutation

relations, but are determined by the equations of motion. For the harmonic oscillator these equations reduce, in our notation, to

$$[Q^2, P] = 2\mathrm{i}Q \qquad [P^2, Q] = -2\mathrm{i}P \tag{6}$$

from which follows

$$\{S, Q\} = 0 = \{S, P\}$$

where $S = \mathrm{i}[Q, P] + 1 = S^*$. Hence (following [15]), S^2 commutes with both Q and P and may therefore be represented as a multiple of the identity operator. We now identify S with a multiple of the reflection operator, $S = -\nu R$ where $\nu \in \mathbb{R}$ since both S, R are Hermitian, to obtain (4).

The algebra (4) is also related to the paraquantization scheme introduced by Green [16] in 1953, for if we define annihilation and creation operators $a, a^\dagger$ in the usual way according to

$$a = \tfrac{1}{\sqrt{2}}(Q + \mathrm{i}P) \qquad a^\dagger = \tfrac{1}{\sqrt{2}}(Q - \mathrm{i}P) \tag{7}$$

then

$$[a, a^\dagger] = 1 + \nu R \qquad \{a, R\} = 0 = \{a^\dagger, R\} \qquad R^2 = 1. \tag{8}$$

These relations in turn imply that $a, a^\dagger$ satisfy the trilinear relations of a paraboson algebra with one degree of freedom:

$$[\{a, a^\dagger\}, a] = -2a \qquad [\{a, a^\dagger\}, a^\dagger] = 2a^\dagger. \tag{9}$$

Paraboson operators and their representations have been investigated extensively, see for example, [17–19] for a discussion of their properties.

We define the vacuum state $|0\rangle$ in the usual way to satisfy

$$a|0\rangle = 0.$$

However, we may choose either an even vacuum $|0\rangle_{\mathrm{e}}$ which satisfies $R|0\rangle_{\mathrm{e}} = |0\rangle_{\mathrm{e}}$ or an odd vacuum $|0\rangle_{\mathrm{o}}$ with $R\,|0\rangle_{\mathrm{o}} = -|0\rangle_{\mathrm{o}}$, which are related by the symmetry (5). An example of such vacua is given by the harmonic oscillator, where we encounter ground states labelled by the angular momentum ℓ, which are even or odd according as ℓ is even or odd (see section 6). From (8) it follows that $aa^\dagger|0\rangle_{\mathrm{e}} = (\nu + 1)|0\rangle_{\mathrm{e}}$, which implies

$$\nu + 1 = {}_{\mathrm{e}}\langle 0|aa^\dagger|0\rangle_{\mathrm{e}} = \|a^\dagger|0\rangle_{\mathrm{e}}\|^2 > 0.$$

Hence, in the space of states built on an even vacuum we are restricted to positive values only of $\nu + 1$, which is the paraboson order, see [17]. Similarly, for states built on the odd vacuum we have $-\nu + 1 > 0$. Although ν can vary continuously, see [19], its values are related to the physical parameters d, ℓ (see equations (13) and (15)) where ℓ is quantized. Within any given model, the parameter ν therefore is assigned a range of values and so should be regarded as an operator which commutes with P, Q, R; alternatively we can regard equations (6) as the defining relations of the algebra, with physical parameters such as d, ℓ appearing within the representations of Q, P.

The extended, or R-deformed, Heisenberg algebra (4) has been considered previously by Vasiliev [13] in connection with higher spin algebras and quantization on the sphere, and also appears in the explicit solution to the N-body Calogero problem [20, 21] for $N = 2$. The solution for general N employs a generalized algebra (see also [22]) which is determined from a covariant derivative which is similar for $N = 2$, but not identical, to that which we use in our representation of the momentum operator, see equation (16). There has also been discussion of the corresponding oscillator systems, which are related to the parabosons mentioned above, in

[23] and the algebra has been interpreted in terms of fractional spin fields [24] and superfields [25] by Plyushchay, who has also investigated the connection with paraboson, parafermion and paragrassmann algebras (see also [26] for summaries). These papers do not, however, interpret the R-deformed Heisenberg algebra in terms of a fractional dimension. Common to many of these papers is the $\mathfrak{sl}_2$ algebra which can be constructed from the R-deformed Heisenberg algebra (4), which we also use in section 7 and which also appears in the construction of Weyl-ordered polynomials [12].

3. Operator representations

The fractional-dimensional interpretation of relations (4) is a consequence of the following representations of P, Q, R, which were found and investigated for dimension $d = 1$ by Yang [15] in 1951, Ohnuki and Kamefuchi [17, 27] and Mukunda *et al* [28] and revisited in a different form by Jing [29]. Similar but more general representations appear also as covariant derivatives in the papers [21, 22] on the Calogero model, amongst others. These representations of P, Q, R were more recently generalized and interpreted in terms of fractional dimensions by Matos-Abiague [30–32], but we consider the following further generalization, in which P depends on two parameters, ν and the dimension d.

Let P, Q, R act in a Hilbert space $\mathfrak{H}$ of complex functions $\psi(x)$ defined on $\mathbb{R}$ with an inner product as defined below in equation (17). The reflection operator R acts according to

$$R\psi(x) = \psi(-x). \tag{10}$$

The position and momentum operators are given by

$$\begin{aligned} Q\psi(x) &= x\psi(x) \\ P\psi(x) &= \left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}\nu}{2}x^{-1}(R-\mu)\right]\psi(x) \end{aligned} \tag{11}$$

which together with R satisfy relations (4) for any $\mu \in \mathbb{R}$. Our aim is to reproduce all the terms in the operator Δ_r given in equation (2), and so we compare Δ_r with $-P^2$. We have

$$-P^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mu\nu}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x^2}\Delta_0'$$

from which we see, by comparison with equation (1), that we may identify the variable x with the radial coordinate r (continued to negative values), and the dimension d according to

$$\mu\nu = d - 1. \tag{12}$$

The operator Δ_0' is given in a factorized form by

$$\Delta_0' = \tfrac{1}{4}(d - 2 - \nu + R)(d - 2 + \nu - R)$$

and has the eigenvalues

$$\tfrac{1}{4}\{(d-1-\nu)(d-3+\nu),\ (d-3-\nu)(d-1+\nu)\}$$

for $R \to 1, -1$, respectively. Each eigenvalue can be expressed in the form

$$-\ell(d + \ell - 2)$$

where ℓ is the angular momentum, provided we identify

$$\nu = d - 1 + 2\ell \qquad \text{or} \qquad \nu = d - 1 + 2(\ell - 1) \tag{13}$$

for $R \to 1, -1$, respectively. Since ν takes different values according to the state on which it acts, we regard ν as an operator which commutes with each of P, Q, R. Let ψ be an eigenfunction of the angular momentum operator Λ, i.e.

$$\Lambda\psi = \ell\psi$$

where ℓ takes the values $\ell = 0, 1, 2, 3, \ldots$. Then we set

$$\nu = d - 2 + 2\Lambda + (-1)^{\Lambda} \tag{14}$$

and since Λ commutes with each of P, Q, R (recalling that Q is represented by the radial coordinate x, which is rotationally invariant) it follows that ν also commutes with P, Q, R, and takes the values shown in (13).

Alternatively, we could also identify ν according to

$$-\nu = d - 1 + 2(\ell - 1) \qquad \text{or} \qquad -\nu = d - 1 + 2\ell \tag{15}$$

for $R \to 1, -1$, respectively, which appears as a manifestation of the symmetry (5). This symmetry is related to the invariance of $\ell(d + \ell - 2)$ under

$$\ell \longrightarrow -\ell - d + 2.$$

Comparing now the operator $-P^2$ and its spectrum with that of Δ_r in (2), we see by a suitable choice of ν that $-P^2$ reproduces all the terms of Δ_r including those involving the fractional dimension d and the angular momentum eigenvalues ℓ.

In previous work [15, 17, 27, 28] the parameter μ was set to zero which, as equation (12) shows, implies $d = 1$. In the papers by Matos-Abiague [30–32] the choice $\mu = 1$ was made in order that the equation $P\psi = 0$ be satisfied for any constant function ψ; this implies $\nu = d - 1$ which in turn restricts the angular momentum to values $\ell = 0$ or $\ell = 1$, as (13) shows. On the other hand, when we compare $-P^2$ and Δ_r as given in equation (2), we see that $\Delta_r\psi = 0$ is satisfied for constant ψ only if $\ell = 0$ or $\ell + d - 2 = 0$. Since this is an unnecessary restriction on ℓ we allow μ to take any real value, and then eliminate μ in favour of ν and the dimension d.

In summary, the momentum operator P is represented by the formal operator

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{i}\nu}{2}x^{-1}R - \frac{\mathrm{i}(d-1)}{2}x^{-1} \tag{16}$$

and is parametrized by ν and $d > 0$. If we regard ν as an operator as shown in (14) then we may identify $-P^2$ and Δ_r as given in (2), i.e. P is the square root of the d-dimensional radial Laplacian $-\Delta_r$. For the purpose of our development, we regard d and ν as independent parameters keeping in mind the identification (13) or (15), and that the independent physical parameters are d and ℓ. If we choose $\nu = 0$ then we have $d = 1$, as (13) shows, under the assumption that ℓ takes quantized values $0, 1, \ldots$. On the other hand, even for one-dimensional models ($d = 1$) we may still retain a nonzero value for the parameter ν which is then directly related to the angular momentum ℓ as given in (13) and (15); this corresponds to the case $\mu = 0$ discussed previously in [15, 17, 28].

4. Domain of *P*

Let us now consider in detail the domain of the operator P given by (16), following [28], where the case $d = 1$ is analysed (and where $\nu = 2\alpha - 1$). We decompose functions ψ into odd and even components, the eigenfunctions of R, according to

$$\psi = \psi_{\mathrm{o}} + \psi_{\mathrm{e}}.$$

The inner product in $\mathfrak{H}$ is given by

$$(\psi, \phi) = \int_{-\infty}^{\infty} |x|^{d-1}\overline{\psi(x)}\phi(x)\,\mathrm{d}x \tag{17}$$

where d is the dimension parameter with $d > 0$, and is relevant not only for models defined on the whole real line (which can be the case for d at or near 1), but also for models defined

on the half-line where we regard x as a radial coordinate which is extended to negative values. We restrict our considerations here to models in which the Hamiltonian H commutes with R, in which case the eigenfunctions are either even or odd and are defined on $\mathbb{R}$.

The inner product may be expressed in the form

$$(\psi, \phi) = 2\int_0^\infty x^{d-1}[\overline{\psi_{\rm e}(x)}\phi_{\rm e}(x) + \overline{\psi_{\rm o}(x)}\phi_{\rm o}(x)]\,{\rm d}x.$$

Elements ψ of $\mathfrak{H}$ must satisfy

$$\begin{aligned}\|\psi\|^2 &= \int_{-\infty}^\infty |x|^{d-1}|\psi(x)|^2\,{\rm d}x \\ &= 2\int_0^\infty x^{d-1}[|\psi_{\rm e}(x)|^2 + |\psi_{\rm o}(x)|^2]\,{\rm d}x < \infty\end{aligned} \tag{18}$$

and therefore also $\|P\psi\|^2 < \infty$. In particular, we require both $\|\psi_{\rm e}\|^2 < \infty$ and $\|\psi_{\rm o}\|^2 < \infty$.

The action of P on functions ψ, which are assumed to be at least once differentiable for $x > 0$, is given by

$$\begin{aligned}(P\psi)_{\rm e}(x) &= -{\rm i}x^{\frac{1}{2}(-\nu-d+1)}\frac{\rm d}{{\rm d}x}\left[x^{-\frac{1}{2}(-\nu-d+1)}\psi_{\rm o}(x)\right] \\ (P\psi)_{\rm o}(x) &= -{\rm i}x^{\frac{1}{2}(\nu-d+1)}\frac{\rm d}{{\rm d}x}\left[x^{-\frac{1}{2}(\nu-d+1)}\psi_{\rm e}(x)\right].\end{aligned} \tag{19}$$

We observe from these expressions that the momentum operator, which we denote by P_d for general d, is related to the momentum operator P_1 for $d = 1$ by a formal similarity transformation $T = x^{\frac{1}{2}(d-1)}$:

$$P_d = T^{-1}P_1T$$

and similarly for Q, R; the corresponding wavefunctions ψ_d, ψ_1 are related according to $\psi_d = T^{-1}\psi_1$. One effect of this transformation is to insert the weight factor x^{d-1} in the inner product as shown in (18). The observation that P_d and P_1 are related by a formal similarity transformation is of mathematical convenience only, since in physical applications ν depends on d as equation (13) shows.

As a consequence of (19), we require $\psi_{\rm e}$, $\psi_{\rm o}$ to behave near $x = 0$ such that

$$\begin{aligned}&\int_0 x^{-\nu}\left|\frac{\rm d}{{\rm d}x}\left[x^{-\frac{1}{2}(-\nu-d+1)}\psi_{\rm o}(x)\right]\right|^2{\rm d}x < \infty \\ &\int_0 x^{\nu}\left|\frac{\rm d}{{\rm d}x}\left[x^{-\frac{1}{2}(\nu-d+1)}\psi_{\rm e}(x)\right]\right|^2{\rm d}x < \infty.\end{aligned}$$

These conditions imply that near $x = 0$

$$\begin{aligned}\psi_{\rm e}(x) &= a_{\rm e}x^{\frac{1}{2}(\nu-d+1)} + O\left(x^{\frac{1}{2}(2-d)+\varepsilon_1}\right) \\ \psi_{\rm o}(x) &= a_{\rm o}x^{\frac{1}{2}(-\nu-d+1)} + O\left(x^{\frac{1}{2}(2-d)+\varepsilon_2}\right)\end{aligned} \tag{20}$$

for constants $a_{\rm e}, a_{\rm o}$ and $\varepsilon_1, \varepsilon_2 > 0$ (the notation in the second term of each right-hand side means that we allow behaviour x^λ where $\lambda > \frac{1}{2}(2-d)$). For $\nu > -1$ these conditions are consistent with the requirement (18) except for $\psi_{\rm o}$ for the case $\nu \geqslant 1$, for which we demand $a_{\rm o} = 0$.

The operators R, Q are Hermitian in the Hilbert space $\mathfrak{H}$ and for P we find

$$(\psi, P\phi) - (P\psi, \phi) = -2{\rm i}[x^{d-1}(\overline{\psi_{\rm e}(x)}\phi_{\rm o}(x) + \overline{\psi_{\rm o}(x)}\phi_{\rm e}(x))]_0^\infty.$$

For $0 < d < 1$ the boundary term at ∞ is zero provided ϕ, ψ are bounded at ∞. For $d \geqslant 1$ the boundary term at ∞ is also zero provided ψ, ϕ vanish at ∞ sufficiently quickly.

In order to consider the boundary terms at $x = 0$ we express each of ψ, ϕ in the form shown in equations (20), for small x:

$$\begin{aligned}
\psi_{\mathrm{e}}(x) &\sim a_{\mathrm{e}} x^{\frac{1}{2}(\nu-d+1)} + a'_{\mathrm{e}} x^{\frac{1}{2}(2-d)+\varepsilon_1} \\
\psi_{\mathrm{o}}(x) &\sim a_{\mathrm{o}} x^{\frac{1}{2}(-\nu-d+1)} + a'_{\mathrm{o}} x^{\frac{1}{2}(2-d)+\varepsilon_2} \\
\phi_{\mathrm{e}}(x) &\sim b_{\mathrm{e}} x^{\frac{1}{2}(\nu-d+1)} + b'_{\mathrm{e}} x^{\frac{1}{2}(2-d)+\varepsilon_3} \\
\phi_{\mathrm{o}}(x) &\sim b_{\mathrm{o}} x^{\frac{1}{2}(-\nu-d+1)} + b'_{\mathrm{o}} x^{\frac{1}{2}(2-d)+\varepsilon_4}
\end{aligned}$$

for constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$. Then

$$\begin{aligned}
(\psi, P\phi) - (P\psi, \phi) &= 2\mathrm{i}\left[\overline{a'_{\mathrm{e}}} b_{\mathrm{o}} x^{\varepsilon_1+\frac{1}{2}(1-\nu)} + \overline{a_{\mathrm{o}}} b'_{\mathrm{e}} x^{\varepsilon_3+\frac{1}{2}(1-\nu)} + \overline{a_{\mathrm{o}}} b_{\mathrm{e}} + \overline{a_{\mathrm{e}}} b_{\mathrm{o}}\right]_{x=0} \\
&= 2\mathrm{i} \begin{cases} 0 & \text{if} \quad \nu \geqslant 1 \\ \overline{a_{\mathrm{o}}} b_{\mathrm{e}} + \overline{a_{\mathrm{e}}} b_{\mathrm{o}} & \text{if} \quad |\nu| < 1 \end{cases}
\end{aligned}$$

where we used $a_{\mathrm{o}} = b_{\mathrm{o}} = 0$ for $\nu \geqslant 1$.

We deduce that for $\nu \geqslant 1$ there is only one way the domain of P in (16) can be chosen which leads to a unique self-adjoint operator: the functions $\psi(x)$ in this domain satisfy $\|\psi\| < \infty$, are once differentiable for $x > 0$, are bounded or vanish sufficiently quickly at ∞ as discussed above, and near $x = 0$ the even and odd components ψ_{e}, ψ_{o} behave as

$$\begin{aligned}
\psi_{\mathrm{e}}(x) &\sim a_{\mathrm{e}} x^{\frac{1}{2}(\nu-d+1)} + a'_{\mathrm{e}} x^{\frac{1}{2}(2-d)+\varepsilon_1} \\
\psi_{\mathrm{o}}(x) &\sim a'_{\mathrm{o}} x^{\frac{1}{2}(2-d)+\varepsilon_2}
\end{aligned} \tag{21}$$

($\varepsilon_1, \varepsilon_2 > 0$) where for $\nu > 1$ the first term in ψ_{e} may be included with the second term. For $|\nu| < 1$ the possible domains $\mathcal{M}_s$ in $\mathfrak{H}$ are parametrized by $s \in \mathbb{R}$, where s is determined by the behaviour of ψ near $x = 0$:

$$\begin{aligned}
\psi_{\mathrm{e}}(x) &\sim a_{\mathrm{e}} x^{\frac{1}{2}(\nu-d+1)} + a'_{\mathrm{e}} x^{\frac{1}{2}(2-d)+\varepsilon_1} \\
\psi_{\mathrm{o}}(x) &\sim \mathrm{i}s a_{\mathrm{e}} x^{\frac{1}{2}(-\nu-d+1)} + a'_{\mathrm{o}} x^{\frac{1}{2}(2-d)+\varepsilon_2}
\end{aligned} \tag{22}$$

i.e. we have $a_{\mathrm{o}} = \mathrm{i}s a_{\mathrm{e}}$. Hence, for all values $\nu > -1$ we have $(\psi, P\phi) = (P\psi, \phi)$ for functions ψ, ϕ in the domain of P. In [28], the case is also discussed where R acts according to $R\psi(x) = -\psi(-x)$; this case may be obtained by means of the symmetry (5) which preserves the inequality $|\nu| < 1$.

So far we have considered only the integrability conditions which are necessary for P to be Hermitian. In actual models, we also require the wavefunctions $\psi(x)$ to satisfy Schrödinger's equation, which means that $\psi(x)$ must be twice differentiable and hence continuous at the origin. Since $\psi_{\mathrm{e}}(x)$ is even, the small x behaviour shown in (20) implies the quantization condition

$$\nu - d + 1 = 4m$$

where m is a non-negative integer. This is consistent with conditions (13) where either $\nu - d + 1 = 2\ell$ for even values of ℓ, or $\nu - d + 1 = 2(\ell - 1)$ for odd values of ℓ. We verify these quantization conditions explicitly in the models considered below.

5. Wave solutions in *d* dimensions and free particles

The eigenfunctions of P reduce to plane waves for $d = 1$ and $\nu = 0$, and in three dimensions to the well-known spherical waves; in general fractional dimensions, the solutions are Bessel functions which exist for any $d > 0$ but with a quantization condition on ℓ.

We solve therefore the equations

$$(P\psi)_{\mathrm{e}}(x) = k\psi_{\mathrm{e}}(x) \qquad (P\psi)_{\mathrm{o}}(x) = k\psi_{\mathrm{o}}(x)$$

where $k \in \mathbb{R}$ is the wavenumber and where the left-hand side is given explicitly in equations (19). Since R does not commute with P the eigenfunctions are neither even nor odd. It is convenient to denote $u(x) = x^{-\frac{1}{2}(\nu-d+1)}\psi_{\mathrm{e}}(x)$, then we find

$$u'' + \frac{\nu u'}{x} + k^2 u = 0.$$

With the help of standard formulae, see for example [33], formula (9.1.52), we obtain the general solution as a linear combination of two independent Bessel functions, in the form

$$\psi_{\mathrm{e}}(x) = ax^{\frac{1}{2}(2-d)}J_{\frac{1}{2}(1-\nu)}(kx) + bx^{\frac{1}{2}(2-d)}Y_{\frac{1}{2}(1-\nu)}(kx)$$

for arbitrary constants a, b. These constants satisfy two linear relations which are determined from $\psi_{\mathrm{e}}(x) = \psi_{\mathrm{e}}(-x)$. We use the analytic continuation formulae for Bessel functions (see [33], formulae (9.1.35) and (9.1.36)) to obtain the following two equations:

$$\begin{aligned} a &= a\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(3-d-\nu)} + 2\mathrm{i}b\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(2-d)}\cos\left[\tfrac{\pi}{2}(1-\nu)\right] \\ b &= b\,\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(1-d+\nu)}. \end{aligned}$$

We choose $b \neq 0$ which implies that we must impose the quantization condition

$$\nu - d + 1 = 4m \tag{23}$$

for some non-negative integer m, which is consistent with equation (13) provided either $\ell = 2m$ for even eigenfunctions of R, or $\ell = 2m+1$ for odd eigenfunctions of R. The alternative choice $b = 0$ is consistent with (15) and corresponds to the convention $R\psi(x) = -\psi(-x)$ rather than (10). We allow only non-negative values of m in order that $\psi_{\mathrm{e}}(x)$ be continuous at the origin, as follows from the small x behaviour of $\psi_{\mathrm{e}}(x)$ (given in equations (26)).

Hence, we find

$$a\cos\frac{\pi\nu}{2} + b\sin\frac{\pi\nu}{2} = 0$$

which enables us to express $\psi_{\mathrm{e}}(x)$ in the form

$$\psi_{\mathrm{e}}(x) = cx^{\frac{1}{2}(2-d)}J_{\frac{1}{2}(-1+\nu)}(kx) \tag{24}$$

where c is any constant. We determine $\psi_{\mathrm{o}}(x)$ from the equation

$$k\psi_{\mathrm{o}}(x) = -\mathrm{i}x^{\frac{1}{2}(\nu-d+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\frac{1}{2}(\nu-d+1)}\psi_{\mathrm{e}}(x)\right]$$

to obtain

$$\psi_{\mathrm{o}}(x) = \mathrm{i}cx^{\frac{1}{2}(2-d)}J_{\frac{1}{2}(1+\nu)}(kx) \tag{25}$$

which is an odd function, as again follows with the help of (23).

For small x we have

$$\begin{aligned} \psi_{\mathrm{e}}(x) &\sim a_{\mathrm{e}}x^{\frac{1}{2}(\nu-d+1)} + a'_{\mathrm{e}}x^{\frac{1}{2}(\nu-d+3)} \\ \psi_{\mathrm{o}}(x) &\sim a'_{\mathrm{o}}x^{\frac{1}{2}(\nu-d+3)} \end{aligned} \tag{26}$$

for constants $a_{\mathrm{e}}, a'_{\mathrm{e}}, a'_{\mathrm{o}}$ which is consistent with the requirements shown in equations (21) and (22) for all $\nu > -1$. We find that the domain of P is $\mathcal{M}_0$ corresponding to $s = 0$ in (22).

The eigenfunctions of P are therefore

$$\psi(k,x) = x^{\frac{1}{2}(2-d)}\left[J_{\frac{1}{2}(-1+\nu)}(kx) + \mathrm{i}J_{\frac{1}{2}(1+\nu)}(kx)\right] \tag{27}$$

which reduces to the expression obtained in [28] for $d = 1$ (see equation (3.20a), where $\alpha = \frac{1}{2}(\nu + 1)$). Since the authors in [28] consider integrability only of the wavefunctions on the half-line $x > 0$, and do not impose differentiability and hence continuity at the origin, they do not obtain the quantization condition (23), which for $d = 1$ reads $\nu = 2\alpha - 1 = 4m$. The result (27) also reduces to the wavefunctions found in [31] (see equation (31)) for a free particle where $\nu = d - 1$, corresponding to $m = 0$ in (23). For $d = 1$, $\nu = 0$ the eigenfunctions of P reduce to the usual plane waves: $\psi(k, x) \propto \mathrm{e}^{\mathrm{i}kx}$.

The free-particle Hamiltonian $H = P^2$ has wavefunctions which are either even or odd and are given by (24) and (25), respectively. In terms of the angular momentum ℓ, we have (substituting $2m = \ell$ for even eigenfunctions and $2m = \ell - 1$ for odd eigenfunctions)

$$\psi_{\mathrm{e},\ell}(k, x) = cx^{\frac{1}{2}(2-d)} J_{\ell-1+\frac{d}{2}}(kx) \qquad \ell = 0, 2, 4, \ldots$$

$$\psi_{\mathrm{o},\ell}(k, x) = \mathrm{i}cx^{\frac{1}{2}(2-d)} J_{\ell-1+\frac{d}{2}}(kx) \qquad \ell = 1, 3, 5, \ldots$$

which we combine into the single formula for all wavefunctions (up to normalization):

$$\psi_\ell(k, x) = x^{\frac{1}{2}(2-d)} J_{\ell-1+\frac{d}{2}}(kx) \qquad \ell = 0, 1, 2, 3, \ldots$$

which is even or odd according as ℓ is even or odd. As expected, these are precisely the eigenfunctions found by Stillinger [4]. For $d = 3$ these functions are, of course, proportional to the spherical Bessel functions $j_\ell(kx)$ which are eigenfunctions of the operator Δ_r for $d = 3$ shown in equation (2), where x denotes the radial coordinate r.

6. Harmonic oscillator in *d* dimensions

The harmonic oscillator and the associated paraboson algebra have been well-studied from an algebraic point of view in [18, 19, 34] amongst others, although not with a fractional-dimensional interpretation. In later work by Jing [29] (in which the paraboson order is denoted by $p = \nu + 1$), the representation (11) of the momentum operator is used but with $\nu = d - 1$, which restricts the possible angular momentum values to $\ell = 0, 1$ as explained above; again, there is no interpretation in terms of fractional dimensions. The paper [32] is also restricted to $\ell = 0, 1$. On the other hand, Stillinger [4] derives the harmonic oscillator eigenfunctions and eigenvalues in fractional dimensions directly by solving (3). We now show that these approaches are all related through the algebras (4) and (8) and the momentum representation (16).

6.1. Algebraic approach

The Hamiltonian for the harmonic oscillator is

$$H = \tfrac{1}{2}(P^2 + Q^2) = \tfrac{1}{2}(aa^\dagger + a^\dagger a) \tag{28}$$

where P, Q satisfy (4) and the paraboson operators $a, a^\dagger$ satisfy relations (8). Since H commutes with R, the eigenstates are either even or odd, and are generated by allowing the creation operator $a^\dagger$ to act on the vacuum, which carries the label ν and may be even or odd.

Considering firstly the case where the vacuum $|0\rangle_{\mathrm{e}}$ is even, the eigenstates are

$$|n\rangle = \mathcal{N}_n^{-\frac{1}{2}} (a^\dagger)^n |0\rangle_{\mathrm{e}} \tag{29}$$

where the normalization is given by

$$\mathcal{N}_n = {}_{\mathrm{e}}\langle 0|a^n (a^\dagger)^n |0\rangle_{\mathrm{e}} = \prod_{k=1}^{n} \left[k + \frac{\nu}{2}(1 - (-1)^k\right]$$

which can be derived with the help of the commutator

$$[a, (a^\dagger)^n] = \left[n + \frac{\nu}{2}(1 - (-1)^n)R\right](a^\dagger)^{n-1} \qquad n \in \mathbb{N}.$$

Since H satisfies $[H, a^\dagger] = a^\dagger$ as follows from (9), H has the eigenvalues

$$E_n = n + \tfrac{1}{2}(\nu + 1) \qquad n = 0, 1, \ldots \tag{30}$$

as is well known (see, for example, [18, 28], where $\alpha = \frac{1}{2}(\nu+1) > 0$ is the lowest eigenvalue). The eigenstates built on the odd vacuum $|0\rangle_o$ are proportional to $(a^\dagger)^n|0\rangle_o$ and in this case H has eigenvalues $n + \frac{1}{2}(-\nu + 1)$, as follows by application of the symmetry (5).

An elegant formulation of the representations of the parabose system in terms of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is given in [28], where $\mathfrak{sl}_2(\mathbb{R})$ is generated by the following operators which are quadratic in the paraboson operators (see also Perelomov [35], chapter 5):

$$L_0 = \tfrac{1}{4}(aa^\dagger + a^\dagger a) \qquad L_+ = \tfrac{1}{2}(a^\dagger)^2 \qquad L_- = \tfrac{1}{2}a^2. \tag{31}$$

The paraboson representations are carried by a direct sum of two $\mathfrak{sl}_2(\mathbb{R})$ representations $\mathcal{D}_\beta$, $0 < \beta < \infty$, where $\beta = \frac{1}{2}\alpha$ or $\frac{1}{2}(\alpha + 1)$ and $\alpha = \frac{1}{2}(\nu + 1)$.

6.2. Explicit wavefunctions

We may relate these algebraic results to properties of wavefunctions in fractional dimensions by representing P, Q, R as shown in equations (16), (11) and (10), respectively. Let us denote the wavefunctions by $\psi_{n,\ell}(x)$. Then these eigenfunctions are either even or odd and satisfy the second-order differential equation (3) with $V(r) = r^2$, that is

$$\tfrac{1}{2}(P^2 + Q^2)\psi_{n,\ell}(x) = \tfrac{1}{2}(-\Delta_r + x^2)\psi_{n,\ell}(x) = \epsilon_n(\ell)\psi_{n,\ell}(x) \tag{32}$$

(identifying the coordinate x with r), which has the following (unnormalized) solutions already given in [4]:

$$\psi_{n,\ell}(x) = L_n^{(\ell-1+\frac{d}{2})}(x^2)x^\ell \,\mathrm{e}^{-\frac{1}{2}x^2}. \tag{33}$$

Here L denotes generalized Laguerre polynomials (see [33], chapter 22, for properties); in particular, we have $L_0^\alpha(x) = 1$ for any values of α, x. The angular momentum ℓ takes the values $\ell = 0, 1, 2, \ldots$ and so $\psi_{n,\ell}(x)$ is even or odd according as ℓ is even or odd. The eigenvalues are given by

$$\epsilon_n(\ell) = 2n + \ell + \frac{d}{2} \tag{34}$$

where n is a non-negative integer. The wavefunctions decrease to zero as $|x| \to \infty$ like $\mathrm{e}^{-\frac{1}{2}x^2}$ and so are normalizable with respect to the inner product (17).

6.3. Excited states

Now let us derive these same eigenfunctions by following the algebraic construction given above in terms of paraboson operators, and also explain why the energy levels in equation (34) depend on $2n$, whereas in the algebraic approach they depend on n (see (30)). The ground (vacuum) state $|0\rangle = \phi^0(x)$ satisfies the equation $a|0\rangle = 0$, which becomes the differential equation

$$(Q + \mathrm{i}P)\phi^0(x) = 0$$

and has the even and odd solutions, respectively,

$$\phi_e^0(x) = x^{\frac{1}{2}(\nu-d+1)}\,\mathrm{e}^{-\frac{1}{2}x^2} \qquad \phi_o^0(x) = x^{\frac{1}{2}(-\nu-d+1)}\,\mathrm{e}^{-\frac{1}{2}x^2}. \tag{35}$$

These solutions accord with the necessary small x behaviour of wavefunctions, required for integrability, as shown in equations (20). Since $\phi^0_{\rm e}(x) = \phi^0_{\rm e}(-x)$ we must have $\nu - d + 1 = 4m$ for some non-negative integer m, as also found for the free-particle Hamiltonian, see (23). For the even wavefunctions, we therefore identify $\ell = 2m$, or substitute $\nu = d - 1 + 2\ell$ directly from (13) or (14), and so $\phi^0_{\rm e}(x) = x^\ell \, {\rm e}^{-\frac{1}{2}x^2}$ where $\ell = 0, 2, 4, \ldots$. The corresponding energy is, according to equation (30),

$$E_0 = \tfrac{1}{2}(\nu + 1) = \ell + \frac{d}{2}.$$

In the case of the odd ground states, which satisfy $R\phi^0_{\rm o}(x) = -\phi^0_{\rm o}(x)$, we identify $-\nu = d - 1 + 2\ell$ as shown in (15) to obtain $\phi^0_{\rm o}(x) = x^\ell \, {\rm e}^{-\frac{1}{2}x^2}$, where now $\ell = 1, 3, 5, \ldots$ for odd functions. The energy is $\frac{1}{2}(-\nu + 1) = \ell + \frac{d}{2}$. In summary, the possible ground states are parametrized by $\ell = 0, 1, 2, \ldots$, and are given by

$$\phi^{0,\ell}(x) = x^\ell \, {\rm e}^{-\frac{1}{2}x^2} \tag{36}$$

and have energy $\ell + \frac{d}{2}$, in agreement with the expressions (33) and (34) for $n = 0$.

We now create the excited states by applying the creation operator $a^\dagger$ to the ground state $|0\rangle = \phi^{0,\ell_0}(x) = x^{\ell_0} \, {\rm e}^{-\frac{1}{2}x^2}$, which has angular momentum ℓ_0. In applying $a^\dagger = \frac{1}{\sqrt{2}}(Q - {\rm i}P)$ we regard ν as the operator given by (14), where the operator Λ acts on the ground state according to $\Lambda\phi^{0,\ell_0} = \ell_0\phi^{0,\ell_0}$. Since Λ commutes with P, Q, R we also have $\Lambda\phi = \ell_0\phi$ for all states ϕ constructed from the ground state, i.e. if ϕ^{0,ℓ_0} carries the angular momentum ℓ_0, then all excited states built on this vacuum also carry the eigenvalue ℓ_0 of Λ. However, ℓ_0 is not necessarily the total angular momentum ℓ which appears in the operator Δ_r given in equation (2), as is clear from the following example. Let us choose ℓ_0 to be even, then $\nu = d - 1 + 2\ell_0$ and the first excited state $\phi^1(x)$, which is an odd function, is given by

$$\phi^1(x) = a^\dagger\phi^{0,\ell_0} = \frac{1}{\sqrt{2}}(Q - {\rm i}P)\phi^{0,\ell_0} \propto x^{\ell_0+1} \, {\rm e}^{-\frac{1}{2}x^2}$$

and according to equation (30) has energy $E_1 = 1 + \frac{1}{2}(\nu + 1) = \ell_0 + 1 + \frac{d}{2}$. By comparison with the solutions (33) and (34) we see that ϕ^1 has the total angular momentum $\ell = \ell_0 + 1$ with a corresponding energy $\epsilon_0(\ell) = \ell + \frac{d}{2}$. We also observe from (36) that $\phi^1(x)$, which is the first excited state built on the even ground state with the even angular momentum ℓ_0, is identical to the odd ground state $\phi^{0,\ell_0+1}(x)$ with the odd angular momentum $\ell_0 + 1$.

The second excited state $\phi^2(x) = (a^\dagger)^2\phi^{0,\ell_0}(x)$, which is even, is proportional to the wavefunction $\psi_{1,\ell_0}(x)$ given in (33) and so has the total angular momentum $\ell = \ell_0$. This state has energy

$$E_2 = 2 + \tfrac{1}{2}(\nu + 1) = 2 + \ell_0 + \frac{d}{2} = \epsilon_1(\ell_0).$$

In general, we have for the nth excited state,

$$\phi^n(x) = (a^\dagger)^n\phi^{0,\ell_0} = \begin{cases} c_n\psi_{\frac{n}{2},\ell_0}(x) & n \text{ even} \\ c'_n\psi_{\frac{n-1}{2},\ell_0+1}(x) & n \text{ odd} \end{cases} \tag{37}$$

where c_n, c'_n are normalization constants. The corresponding energy is

$$E_n = n + \tfrac{1}{2}(\nu + 1) = n + \ell_0 + \frac{d}{2}$$

which agrees with the formula (34) for $\epsilon_n(\ell)$ provided we identify $\ell = \ell_0$ for even n and $\ell = \ell_0 + 1$ for odd n. Specifically,

$$E_{2m} = 2m + \ell_0 + \frac{d}{2} = \epsilon_m(\ell_0)$$
$$E_{2m+1} = 2m + 1 + \ell_0 + \frac{d}{2} = \epsilon_m(\ell_0 + 1)$$

for integers m.

We can also choose an odd ground state by taking ℓ_0 to be odd, and then generate the excited states in the same way as before, except that now we choose $-\nu = d - 1 + 2\ell_0$. The product νR takes the same values as before and hence we generate the same excited states (37).

It is helpful to use the following formula for $a^\dagger$ acting on even functions $\phi_{\rm e}(x)$:

$$a^\dagger \phi_{\rm e}(x) = -\frac{1}{\sqrt{2}}\, {\rm e}^{\frac{1}{2}x^2} x^{\ell_0}\, \frac{\rm d}{{\rm d}x}\left[{\rm e}^{-\frac{1}{2}x^2} x^{-\ell_0}\phi_{\rm e}(x)\right]$$

and also on odd functions $\phi_{\rm o}(x)$:

$$a^\dagger \phi_{\rm o}(x) = -\frac{1}{\sqrt{2}}\, {\rm e}^{\frac{1}{2}x^2} x^{-\ell_0-d+1}\, \frac{\rm d}{{\rm d}x}\left[{\rm e}^{-\frac{1}{2}x^2} x^{\ell_0+d-1}\phi_{\rm o}(x)\right].$$

These formulae lead to alternative expressions for the wavefunctions $\psi_{n,\ell}(x)$.

We observe that it is possible to generate either only even, or only odd, states directly by repeated application of $(a^\dagger)^2$ to the even or odd ground state. For example, all even states are given by

$$\phi^n_{\rm e}(x) = \psi_{n,\ell}(x) = (a^\dagger)^{2n}\phi^{0,\ell}(x)$$

where ℓ is even, and similarly if ℓ is odd. We also point out that it is possible to generate all ground states from the lowest energy vacuum, namely the state $|0\rangle = \phi^{0,0}(x) = {\rm e}^{-\frac{1}{2}x^2}$, by application of $a^\dagger$ in which ν is regarded as the operator

$$\nu = (-1)^\Lambda(d - 1 + 2\Lambda)$$

where Λ now is the full angular momentum operator or, equivalently, the paraboson number operator. Then $a^\dagger|0\rangle \propto x\, {\rm e}^{-\frac{1}{2}x^2}$ and generally $|\ell\rangle = (a^\dagger)^\ell|0\rangle \propto x^\ell\, {\rm e}^{-\frac{1}{2}x^2}$.

For $d = 1$ the results described in this section reduce to the usual results for the harmonic oscillator provided we choose $\ell_0 = 0$ in (37), which implies $\nu = 0$. In this case the generalized Laguerre polynomials in (33) reduce to the form $L^\alpha_n(x^2)$ where $\alpha = \pm\frac{1}{2}$, and are related to the even and odd Hermite polynomials (see [33], formulae (22.5.38)–(22.5.41)). Our results show that we may also quantize the harmonic oscillator for $d = 1$ by choosing nonzero values of $|\nu| = 2\ell_0$; this is the case analysed in [28] from the paraboson viewpoint. The wavefunctions (37) have also been derived in [31] for general d for the special case $\ell_0 = 0$. These results may be extended to the singular oscillator considered by Perelomov [35] (chapter 18) using algebraic methods.

7. Time-dependent harmonic oscillator in *d* dimensions

The approach to quantum mechanics in fractional dimensions described above, which proceeds from the algebra (4) and uses the representations of P, Q, R in equations (16), (11) and (10), respectively, reproduces previously known results but may also be applied to problems not yet studied in fractional dimensions, including those which are amenable to an algebraic approach. One such example is the construction of coherent states in fractional dimensions, which may

be performed using the paraboson operators defined in equation (7) (and considered in [34] for example) but interpreted using the fractional-dimensional momentum operator (16). A second example which we now consider is the solution of time-dependent quantum mechanical models in fractional dimensions. Such models have been extensively investigated in one dimension, using generally the Lewis–Riesenfeld method, see, for example, [36–43], and have various applications such as to ion traps, see [44].

We outline the solution to the time-dependent harmonic oscillator in fractional dimensions, restricting our analysis for simplicity to a constant mass term, although the generalization to a variable mass and the addition of further terms in the Hamiltonian are each possible, due to the Lie algebraic properties of operators quadratric in P, Q. Our aim here is merely to demonstrate that the method of solution using our approach extends to fractional dimensions.

The time-dependent harmonic oscillator in dimension d is defined by the Hamiltonian

$$H(t) = \tfrac{1}{2}(P^2 + \omega(t)^2 Q^2) \tag{38}$$

where the frequency $\omega(t)$ is a given function which depends explicitly on time, and where the operators P, Q satisfy the algebraic relations (4). Since $-P^2$ is identified with the radial Laplacian operator Δ_r by means of the representation (16) our analysis applies to any dimension $d > 0$.

7.1. Construction of the invariant I

In order to solve the time-dependent Schrödinger equation

$$\left[\mathrm{i}\frac{\partial}{\partial t} - H(t)\right]\psi(x,t) = 0$$

we follow the method of Lewis and Riesenfeld [36, 37], and look for a nontrivial Hermitian operator $I(t)$ which satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0. \tag{39}$$

Such an invariant, if it exists, has time-independent eigenvalues λ [36]. After determining the eigenfunctions $\phi_\lambda(x,t)$ of I the method proceeds by constructing solutions of the Schrödinger equation in the form $\psi_\lambda(x,t) = \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}\phi_\lambda(x,t)$ where the phase $\alpha_\lambda(t)$ is given by

$$\frac{\mathrm{d}\alpha_\lambda}{\mathrm{d}t} = \left(\phi_\lambda, \left(\mathrm{i}\frac{\partial}{\partial t} - H\right)\phi_\lambda\right). \tag{40}$$

The general solution of the time-dependent Schrödinger equation is then given by the linear superposition

$$\psi(x,t) = \sum_\lambda c_\lambda \psi_\lambda(x,t) = \sum_\lambda c_\lambda\, \mathrm{e}^{\mathrm{i}\alpha_\lambda(t)}\phi_\lambda(x,t)$$

for constants c_λ.

The first step, therefore, is to solve (39) for $I(t)$ which can be achieved by first expanding $I(t)$ in terms of the following operators:

$$J_0 = \frac{\mathrm{i}}{4}(PQ + QP) \qquad J_+ = -\tfrac{1}{2}Q^2 \qquad J_- = -\tfrac{1}{2}P^2.$$

As a result of equations (6), these operators generate the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0.$$

(It is more convenient to choose the generators in this form rather than the linear combinations given in (31) in terms of paraboson operators, which generate $\mathfrak{sl}_2(\mathbb{R})$.) In terms of the representations (16) and (11) of P, Q we have

$$\begin{aligned} J_- &= \frac{1}{2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(d-1)}{x}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\ell(\ell+d-2)}{x^2}\right] \\ J_+ &= -\frac{1}{2}x^2 \\ J_0 &= \frac{1}{2}\left[x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{d}{2}\right] \end{aligned} \tag{41}$$

where we have replaced ν and R by their eigenvalues as explained in section 3.

The Hamiltonian and $I(t)$ may each be expressed as a linear combination of $\{J_0, J_\pm\}$ with time-dependent coefficients, and the commutator $[H, I]$ is consequently also a linear combination of $\{J_0, J_\pm\}$. Hence, equation (39) implies that these coefficients satisfy a set of coupled ordinary differential equations, which may be solved to obtain $I(t)$ in the following form [36]:

$$I = -\rho^2 J_- - (\rho^{-2} + \dot{\rho}^2)J_+ + 2\mathrm{i}\rho\dot{\rho}J_0 \tag{42}$$

where $\rho(t)$ is a solution of

$$\ddot{\rho} + \omega(t)^2\rho = \frac{1}{\rho^3}. \tag{43}$$

For the time-independent case (constant ω) the solution is $\rho^2 = \omega^{-1}$. Solutions of the nonlinear equation (43) can be obtained, following [45], by solving the following classical equation of motion for the time-dependent harmonic oscillator:

$$\ddot{f} + \omega(t)^2 f = 0.$$

The Wronskian $W(t) = f_1\dot{f}_2 - \dot{f}_1 f_2$ of any two linearly independent solutions f_1, f_2 of this linear equation is a nonzero constant $W = W(t)$, since $\dot{W} = 0$. Then we determine ρ from the equation

$$\rho^2 = c_1 f_1^2 + c_2 f_1 f_2 + c_3 f_2^2$$

where c_1, c_2, c_3 are any constants satisfying

$$W^2\left(4c_1c_3 - c_2^2\right) = 4.$$

7.2. Eigenfunctions of I

The second step is to determine the eigenvalues and eigenfunctions of I, which may be achieved algebraically by defining the time-dependent creation and annihilation operators

$$\begin{aligned} a^\dagger &= \tfrac{1}{\sqrt{2}}[(\rho^{-1} + \mathrm{i}\dot{\rho})Q - \mathrm{i}\rho P] \\ a &= \tfrac{1}{\sqrt{2}}[(\rho^{-1} - \mathrm{i}\dot{\rho})Q + \mathrm{i}\rho P] \end{aligned} \tag{44}$$

from which follows

$$I = \tfrac{1}{2}(a^\dagger a + aa^\dagger)$$

and

$$[a, a^\dagger] = \mathrm{i}[P, Q] = 1 + \nu R \qquad \{a, R\} = 0 = \{a^\dagger, R\}.$$

We find that a, $a^\dagger$ satisfy the same relations (8) as for the time-independent case, and therefore are paraboson operators. The invariant I has the same expression (28) as the Hamiltonian for

the time-independent harmonic oscillator and hence has eigenvalues as shown in (30) for the states (29) constructed on an even vacuum. For the case of an odd vacuum the eigenstates have the eigenvalues $n + \frac{1}{2}(-\nu + 1)$ where n is a non-negative integer.

We deduce that the Lewis–Riesenfeld method generalizes to the fractional-dimensional case without difficulty, essentially because the commutation relations (6) are maintained in fractional dimensions. As a further consequence of these relations the time evolution of $P(t)$, $Q(t)$ is unchanged from the one-dimensional case. We have

$$\dot{P} = \mathrm{i}[H, P] = -\omega(t)^2 Q \qquad \dot{Q} = \mathrm{i}[H, Q] = P$$

and for the creation and annihilation operators we find (following [45])

$$\dot{a}^\dagger = \mathrm{i}\rho(t)^{-2} a^\dagger \qquad \dot{a} = -\mathrm{i}\rho(t)^{-2} a.$$

The solutions are

$$a^\dagger(t) = \mathrm{e}^{\mathrm{i}\Omega(t)} a_0^\dagger \qquad a(t) = \mathrm{e}^{-\mathrm{i}\Omega(t)} a_0$$

where $a_0^\dagger$, a_0 denote the values of $a^\dagger$, a, respectively, at some initial time $t = t_0$, and where

$$\Omega(t) = \int \frac{\mathrm{d}t}{\rho(t)^2}. \tag{45}$$

The time evolution of $P(t)$, $Q(t)$ then follows using equations (44). The invariant I takes the manifestly time-independent form

$$I = \tfrac{1}{2}\left(a_0^\dagger a_0 + a_0 a_0^\dagger\right).$$

In order to determine the explicit dependence on the dimension d of the eigenfunctions ϕ of I we solve the differential equation $I\phi_\lambda = \lambda\phi_\lambda$ where I is given by (42) with $J_\pm$, J_0 represented by equations (41). The possible ground states $|0, t\rangle = \phi^0(x, t)$ are determined by solving $a|0, t\rangle = 0$, which requires us to solve

$$((\rho^{-1} - \mathrm{i}\dot{\rho})Q + \mathrm{i}\rho P)\phi^0 = 0.$$

The solutions, which generalize the ground states (36) for the time-independent case, are (up to a time-dependent normalization):

$$\phi^{0,\ell}(x, t) = x^\ell \exp\left(-\frac{x^2}{2\rho^2}\right) \exp\left(\mathrm{i}\frac{\dot{\rho}x^2}{2\rho}\right) \tag{46}$$

where ℓ is the angular momentum taking values $\ell = 0, 1, 2, \ldots$ as before.

Instead of calculating the general eigenstates $(a^\dagger)^n|0, t\rangle$ of I by direct application of the creation operator to the vacuum, let us perform a unitary transformation on I which enables us to use the previously calculated eigenfunctions (33) of the time-independent harmonic oscillator, and hence determine the complete set of wavefunctions of I. Following [41], and after inspection of the form of the ground states (46), we define the unitary operator

$$U = \exp\left(\frac{\mathrm{i}\dot{\rho}}{\rho} J_+\right) = \exp\left(-\frac{\mathrm{i}\dot{\rho}}{2\rho} Q^2\right).$$

With the help of the formulae

$$U J_0 U^{-1} = J_0 - \mathrm{i}\alpha J_+ \qquad U J_- U^{-1} = J_- + 2\mathrm{i}\alpha J_0 + \alpha^2 J_+$$

where $\alpha = \dot{\rho}/\rho$, we find

$$I' = U I U^{-1} = \tfrac{1}{2}\left(\rho^2 P^2 + \frac{1}{\rho^2} Q^2\right) = \tfrac{1}{2}(P'^2 + Q'^2)$$

where in the last step we defined P', Q' from P, Q as given in equations (16) and (11) by rescaling x according $x = \rho x'$.

The eigenfunctions ϕ' of I' are therefore given by equation (33) with x replaced by $x' = x/\rho$ and hence the eigenfunctions $\phi = U^{-1}\phi'$ of I are, up to normalization, as follows:

$$\phi_{n,\ell}(x,t) = L_n^{(\ell-1+\frac{d}{2})}\left(\frac{x^2}{\rho^2}\right) x^\ell \rho^{-\frac{1}{2}(d+2\ell)} \exp\left(-\frac{x^2}{2\rho^2}\right) \exp\left(\mathrm{i}\frac{\dot\rho x^2}{2\rho}\right) \tag{47}$$

where, as before, ℓ is the angular momentum, L denotes generalized Laguerre polynomials and the non-negative integer n denotes the energy levels. The ground states (46) correspond to the value $n = 0$. We have included the factor $\rho^{-\frac{1}{2}(d+2\ell)}$ in (47) in order that the normalization be time independent. This follows since we have

$$\|\phi_{n,\ell}\|^2 = \int_{-\infty}^{\infty} |x|^{d-1} |\phi_{n,\ell}(x,t)|^2 \,\mathrm{d}x$$

in which we change the integration variable by substituting $x = \rho x'$. We observe that $|\phi_{n,\ell}(\rho x', t)|^2$ depends on t only through the factor ρ^{-d} which is in turn cancelled by ρ^d arising from the change of variable in the measure $|x|^{d-1}\,\mathrm{d}x$, and hence $\|\phi_{n,\ell}\|$ is independent of t.

The eigenvalues $\lambda_n = \epsilon_n(\ell)$ of I are given by (34), namely

$$\epsilon_n(\ell) = 2n + \ell + \frac{d}{2}.$$

This is consistent with the energy levels $E_n = n + \frac{1}{2}(\nu + 1)$ found algebraically, as discussed for the time-independent harmonic oscillator in section 6.

The calculation of the phases $\alpha_n(t)$ as shown in (40) proceeds as in the one-dimensional case (see, for example, [36, 41]); we find $\dot\alpha_n(t) = -\left(2n + \ell + \frac{d}{2}\right)\big/\rho(t)^2$ and hence

$$\alpha_n(t) = -\left(2n + \ell + \frac{d}{2}\right)\Omega(t)$$

where $\Omega(t)$ is defined in (45). Finally, we obtain the following solutions of the time-dependent harmonic oscillator, as defined by the Hamiltonian (38):

$$\psi_{n,\ell}(x,t) = c_n L_n^{(\ell-1+\frac{d}{2})}\left(\frac{x^2}{\rho^2}\right) x^\ell \rho^{-\frac{1}{2}(d+2\ell)} \times \exp\left(-\frac{x^2}{2\rho^2}\right) \exp\left(\mathrm{i}\frac{\dot\rho x^2}{2\rho}\right) \exp\left[-\mathrm{i}\left(2n + \ell + \frac{d}{2}\right)\Omega(t)\right]. \tag{48}$$

The normalization constant c_n of these wavefunctions depends on d and ℓ, but is independent of time, by construction. For constant $\rho^2 = \omega^{-1}$ these wavefunctions reduce to the previous expressions shown in equation (33), with the time evolution determined by the factor $\exp(-\mathrm{i}\omega^2\epsilon_n(\ell)t)$. In general, the wavefunctions (48) are well defined for any $d > 0$ and any $\ell = 0, 1, 2, \ldots$.

8. Conclusions

We have developed an approach to quantum mechanics in fractional dimensions which is algebraic to the extent that it takes relations (4), which depend on ν, as a starting point, but in which the fundamental physical parameters d and ℓ appear within the representation of P. We have shown that the algebraic properties of the system extend to properties of creation and annihilation operators which we may view as paraboson operators with one degree of freedom. We have analysed the domain of P and determined conditions for which P is Hermitian, and then found the eigenfunctions of P and consequently also the free-particle wavefunctions. We have demonstrated that the quantization of the angular momentum ℓ

follows from the required continuity of the wavefunctions at the origin. We have solved the standard harmonic oscillator in the dimension d (reproducing results found by Stillinger and others for special cases), but have also included general values of the angular momentum ℓ. We have demonstrated the consistency of the algebraic and analytic approaches which are fundamental to the construction of coherent and squeezed states. Finally, we have used our approach to solve the time-dependent harmonic oscillator in fractional dimensions using the method of Lewis and Riesenfeld. We have calculated the explicit wavefunctions for any dimension d and for any (integer) values of ℓ which, to our knowledge, have not previously been derived. Our approach is applicable to other models and further generalizations also, and work on this is in progress.

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